International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Coverings of [Mon] and Minimal Orthomodular Lattices

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Received July 4, 1997

A finite, nonmodular orthomodular lattice (OML) T is called minimal if all its proper subOMLs are modular. For a finite, nonmodular OML T, T minimal is equivalent to the equational class $[T]$, generated by T, covers the equational class [MOn] for some n. The main result of this paper is that there exist infinitely many minimal OMLs. They are obtained from quadratic spaces on finite fields. The automorphism groups of such OMLs are given.

1. INTRODUCTION AND REVIEW

We use the abreviation OML for orthomodular lattice. The general reference for such lattices is Kalmback (1982).

Let T be a finite, nonmodular OML; T is called minimal if all the proper subOMLs of T are modular.

A first study of these structures was presented at the Quantum Structures '94 Conference in Prague. The following results can be found in Carréga (1995):

1. If a finite, nonmodular OML T is minimal, T is irreducible or $T \sim T' \times U$, where T' is an irreducible minimal OML and $U = \{0,1\}$ is the two-element Boolean algebra.
2. The simplest example of minimal OML, called T_1, is the horizontal sum of two Boolean algebras with two and three atoms. Except for T_1, the other irreducible minimal OMLs have only blocks with three atoms.
3. If T is a finite, nonmodular OML, T is minimal if and only if the equational class $[T]$ generated by T covers the equational class

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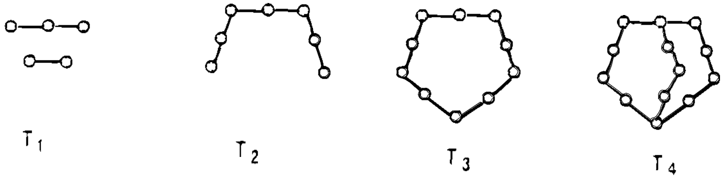

Fig. 1.

[MOn] generated by MOn for some $n \geq 2$. (MOn denotes the modular ortholattice, horizontal sum of n four-element Boolean algebras).

4. With Richard Greechie, we found the following minimal OMLs, given here by their Greechie diagrams (Figs. 1 and 2).

2. PRONGS AND AMALGAMATION

1. In the list of the T_i, $1 \leq i \leq 8$, we see that T_6 is obtained from T_5 by pasting a kind of tail.

Actually this tail, called a prong by R. Greechie, is a modular OML isomorphic to MO3 $\times$ U. There is also prong with two blocks isomorphic to MO2 $\times$ U; for example, T_3 is obtained from T_2, and T_4 from T_3 by pasting a prong with two blocks.

Proposition 1. The only irreducible minimal OMLs obtained from another one by pasting a prong are T_3, T_4, T_6, T_8.

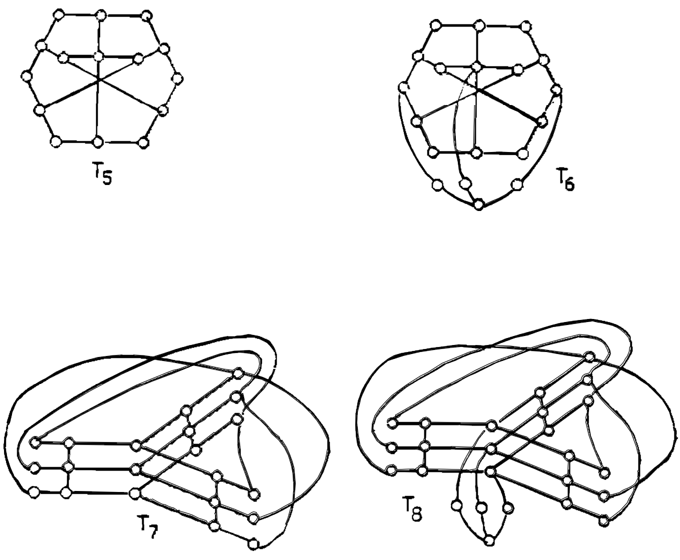

Fig. 2.

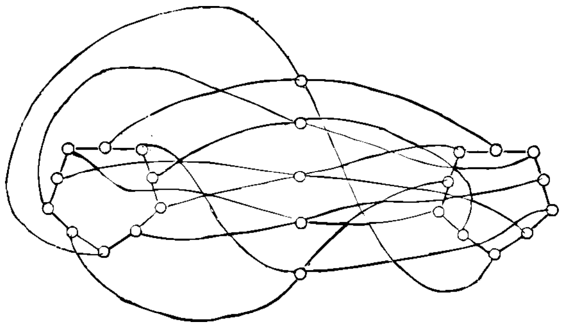

Fig. 3.

2. R. Greechie found a new way to built new minimal OMLs from others; he called his construction amalgamation.

(a) Take two copies of T_3, take five new atoms and new blocks through these atoms for linking each atom of a copy to an atom of the other copy. So we obtain a new minimal OML called T_9 (Fig. 3). For T_9 the linking associates two atoms of the same type; using a linking which associates atoms of different type we obtain T'_9 (Fig. 4).

(b) The minimal OML, called T_{10}, is obtained by amalgamation of two copies of T_5. For performing the linking in a good way, R. Greechie uses a dispertion-free state of T_5. This minimal OML is interesting because it is self-dual. It has 35 atoms and 35 blocks and there exists a duality between atoms and blocks (Fig. 5).

Figure 6 gives a summary of the covering problem concerning the minimal OMLs given above.

3. MINIMAL OMLs FROM VECTOR SPACES

Following Flachsmeyer (1995), René Mayet had the idea to obtain minimal OMLs in the following way:

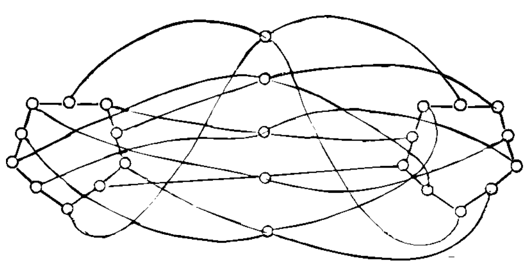

Fig. 4.

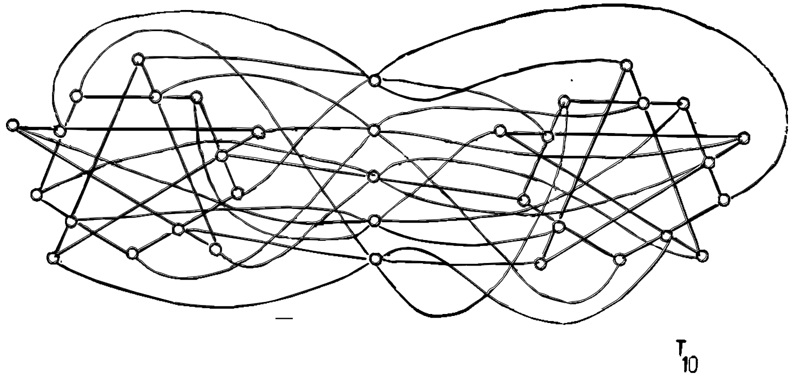

Fig. 5.

Let K be a finite field; the cardinality of K is denoted by q.

Take E, the three-dimensional vector-space over K;

L, the modular lattice of all the subspaces of E;

Q, the canonical quadratic form over E, and $\langle\ ,\ \rangle$ the corresponding inner product:

$$Q(u) = x^2 + y^2 + z^2$$

$$\langle u, u' \rangle = xx' + yy' + zz' \qquad \text{for} \quad u = (x, y, z), u' = (x', y', z')$$

For M in L, denote $M^{\perp} = \{u \in E / \forall v \in M, \langle u, v \rangle = 0\}$.

An isotropic vector is a vector $u \neq 0$ such that $Q(u) = 0$. An isotropic subspace is a subspace M such that the restriction of Q to M is degenerate. The isotropic subspaces are the subspaces of the form $M = K\omega$ or $M = (K\omega)^{\perp}$, where ω is an isotropic vector.

Proposition 2. Let T be the set of all the nonisotropic subspaces of E; then $(T, \subset, \perp)$ is an OML.

In the following, we have to separate two cases according to the characteristic of the field K. With R. Mayet, after a long proof, we prove the following theorem. This theorem is our main result; it provides infinitely many minimal OMLs.

Theorem 1. If the characteristic of the field K is 2, then:

(a) For $q = 2$, $T = T_1$.

(b) For $q = 2^n$, $n \geq 2$, T is the horizontal sum of an OML T' with a four-element Boolean algebra. For every n prime, T' is a minimal OML and $[T']$ covers $[\mathrm{MO}(q-1)]$.

Example: For $q = 4$, $T' = T_5$.

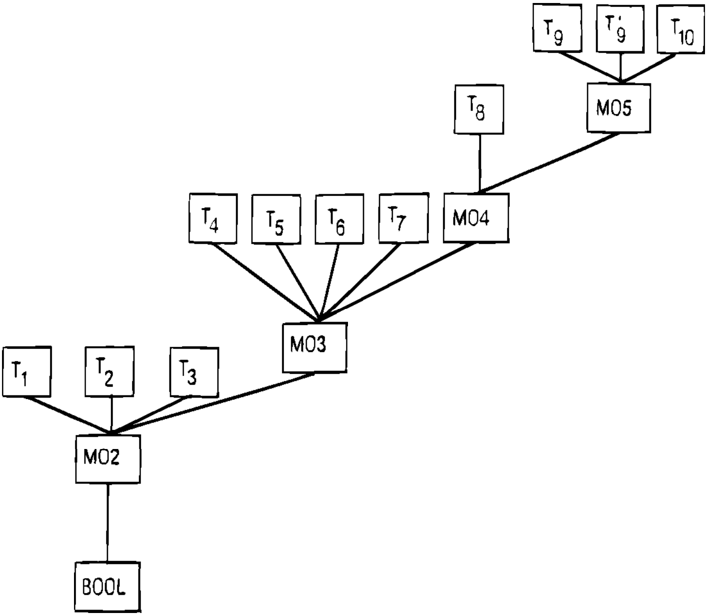

Fig. 6.

If the characteristic of the field K is different from 2, $q = p^n$ with p prime $\neq 2$.

In this case, $n = 1$ is necessary to have a minimal OML.

The case $q = p = 3$ is a special case where T is not a minimal OML, but for $q = 5, 7, 11, 13$ we proved with a computer program that T is minimal and covers [MOq]. We expect a general result for all q prime ≥ 5.

Example: For $q = 5$, $T = T_9$.

The following theorem gives the automorphism groups of the OMLs T and T' introduced by Proposition 2 and theorem 1. Denote by $O_s(E)$ the semiorthogonal group of the quadratic space E. An element f of $O_s(E)$ is a σ-linear bijection f:E $\rightarrow E$ for some $\sigma \in \mathrm{Aut}(K)$ such that

$$\forall u \quad \forall v \quad \langle f(u), f(v)\rangle = \sigma(\langle u, v\rangle)$$

Theorem 2. (a) If $q = 2^n$, $n \geq 2$, $\mathrm{Aut}(T') \sim O_s(E)$ and $|\mathrm{Aut}\ (T')| = nq(q^2 - 1)$.

(b) If $q = p^n$, p prime $\neq 2$ and 3, $\mathrm{Aut}(T) \sim O_s(E)/\{\pm\, id_\varepsilon\}$ and $|\mathrm{Aut}(T)| = nq\ (q^2 - 1)$.

We have in preparation a synthetic paper with complete proofs, in collaboration with Richard Greechie (Ruston) and René Mayet (Lyon).

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